

Spin Operators for Massive Particles

Taeseung Choi^{1,2,*} and Sam Young Cho^{3,†}

¹*Division of General Education, Seoul Women's University, Seoul 139-774, Korea*

²*School of Computational Sciences, Korea Institute for Advanced Study, Seoul 130-012, Korea*

³*Centre for Modern Physics and Department of Physics,
Chongqing University, Chongqing 400044, China*

How to define a proper relativistic spin operator, as a long-standing problem, has by now become a central task for providing proper concepts and applications of spin in relativistic and non-relativistic quantum mechanics as well as solving emergent inconsistencies in rapidly developing research areas. We rigorously *derive* a relativistic spin operator for an arbitrary spin massive particle on the two requirements that a proper spin operator should satisfy (i) the $\mathfrak{su}(2)$ algebra and (ii) the Lorentz-transformation properties as a second-rank spin tensor. These requirements lead to two spin operators, properly giving the second Casimir invariant operator in the Poincaré (inhomogeneous Lorentz) group, that provide the two inequivalent representations of Poincaré group. We find that the two inequivalent representations are the left-handed and the right-handed representations. Each of the two spin operators generates a Wigner little group whose representation space is composed of spin- s spin states. In the case that the Poincaré group is extended by parity, only nonchiral (s, s) representations and direct-sum $(s, s') \oplus (s', s)$ representations are allowed. In the $(1/2, 0) \oplus (0, 1/2)$ representation, we rederive the covariant Dirac equation by using the covariant parity operator defined by the two spin operators. This derivation deepens our understanding how the Dirac equation describes the spin-1/2 massive relativistic particle successfully. For a single-spin massive particle satisfying the Dirac-like equations in the $(s, 0) \oplus (0, s)$ representation, it is found that, as a good observable, its spin current are conserved, which implies that the relativistic spin and the relativistic spin current are applicable to understand inconsistent phenomena emerging in spintronics. Furthermore, we show that the helicity operator for the massive particle in an arbitrary frame can be represented by the one in the particle rest frame.

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Introduction.— Spin of a massive particle (e.g. electron) has become a very familiar physical quantity [1] since it was introduced to explain the broadening of the sodium D-lines observed by Zeeman in 1897 [2] and the splitting of the silver beam observed by Stern and Gerlach in 1922 [3]. Still its crucial roles have been revealed in a wide range of (relativistic) quantum phenomena in currently developing research topics such as (relativistic) quantum information and communication, spintronics and spin Hall effects. However, for instance, properly defining reduced spin state (spin entropy) [4], spin current operator [5], and dealing with spin-dependent forces have been challenged to solve some controversial issues in such research areas. In addition, there is the so-called proton spin crisis [6] where it seems that the nucleon spin is not simply made up by the quark spins, contrary to our desirous belief. Basically, such puzzling inconsistencies may come from a vulnerable understanding of the origin of spin in relativistic as well as non-relativistic quantum mechanics because an appropriate spin operator has not been uncovered [7].

Since the birth of Dirac theory in 1928 [8], actually, many proposals have been made for a proper relativistic spin [9–20]. However, such proposed relativistic spin operators with different assumptions have not been reached a consensus for a proper spin operator because each proposed relativistic spin operator has not been shown to have a sufficient justification to implement physical re-

quirements [21–23]. Even the spin interactions being widely studied in nonrelativistic quantum mechanics, to our best knowledge, still do not have a concrete theoretical verification for its origin from first principles, though it has been thought of appropriate for the nonrelativistic spin (the Pauli spin operator). More systematic derivation in defining a proper spin operator has been then required to understand more crucial physical properties of spin in solving such emergent inconsistencies. However, this undoubtedly challenging fundamental problem has remained as an inherently obstacle as ever in understanding the origin of spin and its associated emerging physics. Therefore, a properly defined spin operator might shine much light on resolving such emergent challenging issues.

As was shown by Wigner in 1939 [24], in fact, spin is naturally introduced through the little group in the unitary representation of the Poincaré (inhomogeneous Lorentz) group. Essentially, the two invariant Casimir operators of the Poincaré group, i.e., $P^\mu P_\mu = m^2$ and $W^\mu W_\mu = -m^2 s(s+1)$, are known to give the rest mass m and the spin s of the particle, respectively, where the Pauli-Lubanski (PL) vector is defined as $W^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma$ with a four-dimensional Levi-Civita $\epsilon_{\mu\nu\rho\sigma}$ (we set $\epsilon_{0123} = \epsilon^{1230} = 1$), the generators of the Lorentz group $J^{\mu\nu}$, and the generators of translations P^μ . Here, Einstein summation convention is used for the Greek indexes $\mu \in \{0, 1, 2, 3\}$ and will be also used for Latin indexes $k \in \{1, 2, 3\}$, unless otherwise specifically stated.

We will omit the word ‘operator’ freely, e.g., PL vector instead of PL vector operator, because the context will clarify the usage. The metric tensor $g_{\mu\nu} = \text{diag}(+, -, -, -)$ will be used. As is known, however, the spatial components of the PL vector W^k cannot be a spin three-vector because they do not satisfy even the basic requirement of a proper spin operator, i.e., the $\mathfrak{su}(2)$ algebra, which obscures the fundamental nature of spin and has led many different definitions for a proper spin operator. However, the Lorentz-invariant square of the PL vector offers a way to reach the proper spin operator. The second Casimir invariant actually implies that the square of (relativistic) spin three-vector \mathbf{S} is well-defined in the Poincaré group as

$$\mathbf{S} \cdot \mathbf{S} = \frac{1}{m^2} W^\mu W_\mu. \quad (1)$$

Practically, this fact allows us to derive a proper spin three-vector operator \mathbf{S} as a linear combination of PL vectors in arbitrary frames from the physical requirements and to investigate its fundamental properties systematically.

Derivation of spin three-vector.— (i) *The spin three-vector is a linear combination of PL vectors.* Let us start with a general form of spin three-vector as a linear combination of PL vectors, i.e., the k -component of the \mathbf{S} :

$$S^k = a_{k,0}W^0 + a_{k,k}W^k + a_{k,m \neq k}W^m \equiv a_{k,\mu}W^\mu, \quad (2)$$

where the index k in $a_{k,k}$ is not considered as repeated. The momentum and the spin (an eigenvalue of the S^k), denoting the two Casimir invariants, are expected to label the representation of the Poincaré group. This requires that the S^k should commute with the momentum operator P^μ . Thus, the coefficient $a_{k,\mu}$ is a function of complex numbers and the momentum operators P^μ but it is not a function of the Lorentz generators $J_{\mu\nu}$.

(ii) *The spin three-vector is a spatial three-dimensional vector.* In classical physics, a spatial angular momentum vector and the total angular momentum vector are a spatial three-vector. It is then natural that a spin angular momentum vector is also regarded as a spatial three-dimensional vector. Thus, the spin three-vector transforming as a three-dimensional vector under a spatial rotation should satisfy

$$[J^j, S^k] = i\epsilon_{jkl}S^l, \quad (3)$$

where J^j is the rotation generator around the axis \hat{x}^j and ϵ_{jkl} is a three-dimensional Levi-Civita with $\epsilon_{123} = 1$. Let us substitute Eq. (2) into the commutation relation in Eq. (3). Equation (3) becomes

$$[J^j, a_{k,\mu}]W^\mu + i\epsilon_{jln}a_{k,l}W^n = i\epsilon_{jkl}a_{l,\nu}W^\nu \quad (4)$$

by using $J^i = \epsilon_{ijk}J^{jk}/2$ and $[J^{\lambda\mu}, W^\nu] = i(g^{\mu\nu}W^\lambda - g^{\lambda\nu}W^\mu)$. Since all W^μ terms are linearly independent,

the coefficients $a_{k,\mu}$ from Eq. (4) have the conditions:

$$[J^j, a_{k,0}] = i\epsilon_{jkl}a_{l,0} \quad \text{for } W^0, \quad (5a)$$

$$[J^j, a_{k,k}] + i\epsilon_{jlk}a_{k,l} = i\epsilon_{jkl}a_{l,k} \quad \text{for } W^k, \quad (5b)$$

$$[J^j, a_{k,m \neq k}] + i\epsilon_{jlm}a_{k,l} = i\epsilon_{jkl}a_{l,m \neq k} \quad \text{for } W^m. \quad (5c)$$

As a function of the momentum operators, the coefficient $a_{k,0}$ in Eq. (5a) is a function of P^k and P^0 because for $j = k$, $[J^k, a_{k,0}] = 0$ is guaranteed from $[J^k, P^0] = 0$ and $[J^k, P^k] = 0$ in the commutation relation between $J^{\mu\nu}$ and P^ρ , i.e., $[J^{\mu\nu}, P^\rho] = i(g^{\nu\rho}P^\mu - g^{\mu\rho}P^\nu)$. In order to satisfy Eq. (5a) for $j \neq k$, also, $a_{k,0}$ should be a linear function of P^k because if it is a quadratic and more higher order function of P^k then the left-hand side of Eq. (5a) becomes zero, but the right-handed side of Eq. (5a) cannot be zero with general momenta. Then, the coefficient $a_{k,0}$ of the term W^0 can be written as

$$a_{k,0} = f_0(P^0)P^k, \quad (6)$$

where $f_0(P^0)$ is a function of P^0 .

Equation (5c) becomes $[J^k, a_{k,m \neq k}] = i\epsilon_{kml}a_{k,l}$ for $j = k$ and $[J^m, a_{k,m \neq k}] = i\epsilon_{mkl}a_{k,l}$ for $j = m$. This implies that the non-commuting part of the operator $a_{k,m \neq k}$ transforms as the m - or k -component of a three-vector under a rotation. In three-dimension, only two types of vectors are possible. One is an ordinary vector \mathbf{P} , the other is a pseudovector $\mathbf{P} \times \mathbf{C}$ with a constant vector \mathbf{C} . To satisfy Eq. (5c), then, the $a_{k,m \neq k}$ is expressed as

$$a_{k,m \neq k} = f_2(P^0)P^kP^m + f_3(P^0)\epsilon_{kml}P^l, \quad (7)$$

where $f_2(P^0)$ and $f_3(P^0)$ are functions of P^0 .

The coefficient $a_{k,k}$ in Eq. (5b) is a function of P^k and P^0 because $a_{k,k}$ commutes with J^k for $j = k$. For $j \neq k$, furthermore, Eq. (5b) becomes $[J^j, a_{k,k}] = 0$ by using the coefficient $a_{k,m \neq k}$ in Eq. (7). At this stage, then, the coefficient $a_{k,k}$ is not specified more. However, for $j \neq k \neq m$, Eq. (5c) can be $[J^j, a_{k,m \neq k}] + i\epsilon_{jkm}a_{k,k} = i\epsilon_{jkm}a_{m,m}$. Satisfying this condition, $a_{k,k}$ can have a form of $f_1(P^0)$ or $f_2(P^0)P^kP^k$. The coefficient $a_{k,k}$ can then be written as

$$a_{k,k} = f_1(P^0) + f_2(P^0)P^kP^k, \quad (8)$$

where $f_1(P^0)$ is a function of P^0 . Consequently, as a three-dimensional vector satisfying Eq. (3), Eq. (2) can be rewritten in terms of a more specific form of the coefficients $a_{k,\mu}$:

$$S^k = f_0(P^0)P^kW^0 + f_1(P^0)W^k + f_2(P^0)P^kP^kW^k + f_3(P^0)\epsilon_{kml}P^lW^m. \quad (9)$$

(iii) *The spin three-vector operators are the generators of SU(2) group* such that they should satisfy the $\mathfrak{su}(2)$ algebra, i.e., the commutation relations,

$$[S^i, S^j] = i\epsilon_{ijk}S^k. \quad (10)$$

Let us put Eq. (9) into the commutation relation in Eq. (10). By using the commutation relations $[W^0, W^k] = i\epsilon_{klm}W^lP^m$ and $[W^i, W^m] = i\epsilon_{iml}(W^lP^0 - W^0P^l)$, three equations are obtained as

$$f_0 + f_2P^0 = -f_0f_1P^0 - f_1^2 + m^2f_3^2 - f_1f_2P_0^2, \quad (11a)$$

$$f_1 = (f_0 + f_2) f_1 (P_0^2 - m^2) + f_1^2 P^0, \quad (11b)$$

$$f_3 = (f_0 + f_2) f_3 (P_0^2 - m^2) + f_1f_3P^0. \quad (11c)$$

From Eqs. (11a), (11b), and (11c), however, f 's cannot be determined because the three equations have the four variables, which means that infinitely many solutions are possible with respect to f 's. Moreover, Eqs. (11b) and (11c) are not independent each other.

(iv) *The spin three-vector transforms as a $k0$ -component of a second-rank tensor under the Lorentz transformation.* To specify f 's more, we consider the fact that the angular momentum three vectors are obtained from the second-rank tensors. In the same manner of the relation between the angular momentum tensor and the angular momentum three-vector, the spin three-vector is denoted by using the spatial components of a spin tensor $S^{\mu\nu}$, i.e.,

$$S^k = \frac{1}{2}\epsilon_{klm}S_{lm}. \quad (12)$$

Crucially, S^k is the $k0$ -component of the dual spin tensor $*S^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}S_{\rho\sigma}$, i.e.,

$$*S^{k0} = S^k \quad (13)$$

because $*S^{k0} = \frac{1}{2}\epsilon^{k0lm}S_{lm} = \frac{1}{2}\epsilon_{klm}S_{lm}$. Hence, Eqs. (12) and (13) imply that S^k should be transformed as a $k0$ -component of a second-rank tensor for a Lorentz transformation. In fact, this tensorial requirement is a generalization of the spatial three-vector condition in Eq. (3) because if a Lorentz transformation becomes a spatial rotation then the tensorial requirement reduces to the spatial three-vector condition in (ii). As well as the spatial three-vector condition in Eq. (3), thus, the tensorial requirement gives the additional condition.

Then, f 's given in Eqs. (6), (7), and (8) from the spatial three-vector condition can be specified more by the additional condition as follows. Under a Lorentz transformation, f_1 should be linearly proportional to P^0 , i.e., $f_1(P^0) = bP^0$, to make the term of $f_1(P^0)W^k$ transforming like a $k0$ -component of the tensor, while f_0 and f_3 should be constant (scalar), i.e., $f_0(P^0) = a$ and $f_3(P^0) = c$, because, for instance, the terms of $P^k W^0$ and $\epsilon_{kml}P^l W^m = \epsilon_{0kml}P^l W^m$ already transform like a $k0$ -component, where b and c are constant (scalar under the Lorentz transformation). However, the term of $f_2(P^0)P^k P^n W^n$ is converted to a form of $f_2(P^0)P^k W^0 P^0$ by using $W^\mu P_\mu = 0$. This implies that actually this term is a $k00$ -component of a third-rank tensor. Thus, to satisfy the tensorial property of the

second-rank spin tensor, one has to set $f_2(P^0) = 0$. Consequently, Eq. (9) can be rewritten as a more specific form:

$$S^k = a P^k W^0 + b P^0 W^k + c \epsilon_{kml}P^l W^m. \quad (14)$$

On substituting Eq. (14) into Eq. (10), Eqs. (11a), (11b), and (11c) become, respectively,

$$a = -a b P_0^2 - b^2 P_0^2 + m^2 c^2, \quad (15a)$$

$$b = a b (P_0^2 - m^2) + b^2 P_0^2, \quad (15b)$$

$$c = a c (P_0^2 - m^2) + b c P_0^2. \quad (15c)$$

For an arbitrary P^0 , the three equalities in Eqs. (15a), (15b) and (15c) should hold, which means that both the coefficients of P^0 and the constant terms in the equalities should be zero. To determine the three constants a , b , and c , then, we obtain the six conditions:

$$a(a+b) = 0 \quad \text{and} \quad a - m^2 c^2 = 0, \quad (16a)$$

$$b(a+b) = 0 \quad \text{and} \quad b(1+m^2 a) = 0, \quad (16b)$$

$$c(a+b) = 0 \quad \text{and} \quad c(1+m^2 a) = 0. \quad (16c)$$

These six conditions clearly show that if one of the three constants is zero then all of the three constants become zero. Hence, all of them should be nonzero and then the six conditions reduce to the three conditions:

$$a + b = 0, \quad (17a)$$

$$1 + m^2 a = 0, \quad (17b)$$

$$a - m^2 c^2 = 0. \quad (17c)$$

One can obtain the two sets of the three constants as

$$a = -\frac{1}{m^2}, \quad b = \frac{1}{m^2}, \quad \text{and} \quad c = \pm \frac{i}{m^2}. \quad (18)$$

Resultantly, we obtain the two spin three-vectors as

$$S_\pm^k = \frac{1}{m^2} (P^0 W^k - P^k W^0) \pm \frac{i}{m^2} \epsilon_{kml} P^l W^m. \quad (19)$$

Note that in deriving the two spin operators in Eq. (19), we used the minimal conditions, i.e., the $\mathfrak{su}(2)$ algebraic requirement in (iii) and the tensorial requirement in (iv), because the tensorial requirement includes the spatial three-vector condition in (ii). It should be also noted that the two spin operators S_\pm^k in Eq. (19) are valid for all reference frames.

(vi) *The spin three-vectors S_\pm^k give the Casimir invariant in the Poincaré group as proper spin operators.* The derived two spin operators S_\pm^k must give the second Casimir invariant $W^\mu W_\mu$ in order that they are the proper spin operator in the Poincaré group. By using Eq. (19) for the spin three-vectors, straightforwardly, one can show $S_\pm^k S_\pm^k = -W^\mu W_\mu / m^2$, where the minus sign comes from the calculation $S_\pm^k S_\pm^k (= -S_\pm^k S_{k,\pm})$. In fact, the derived two spin operators offer the same Casimir operator

of the Poincaré algebra, i.e., $S_+^k S_+^k = S_-^k S_-^k$. The operators $S_+^k S_+^k$ and $S_-^k S_-^k$ have the eigenvalues $s_+(s_+ + 1)$ and $s_-(s_- + 1)$, respectively, because they are the generators of SU(2) groups. The two spin operators do not commute each other in an arbitrary frame, i.e., $[S_+^i, S_-^j] \neq 0$ and cannot be mapped to each other by a similarity transformation. Consequently, there are two inequivalent representations for a massive particle with mass m and spin s_\pm , each of which is labelled by either the eigenvalues of $\{P^\mu, S_+^k\}$ or $\{P^\mu, S_-^k\}$. In the Poincaré group, actually, the two representations are associated with the transformation properties of particles states under the Lorentz boost transformations, i.e., particle's handedness. It will become clear in detailed discussions of the following sections.

Properties of the two spin operators.— Note that we have derived the spin operators, presenting the Poincaré group, in Eq. (19). Let us then study some important properties of the two spin operators such as a general representation, a little group, handedness, and conservation of spin and spin currents. Parity operation makes significant roles of the two spin operators clear and its covariant definition is obtained by using the two spin operators. Using the covariant parity operator on the left-handed and the right-handed eigenstates reproduces the covariant free Dirac equation. This implies that the free Dirac equation already contains the context of the relativistic spin.

(vii) *Two spins S_\pm^k naturally imply handedness and give nonchiral (s, s) representations and direct sum $(s, s') \oplus (s', s)$ representations for the Poincaré group extended by parity.* As was discussed in (vi), the irreducible representations of the Poincaré group can be classified into the two inequivalent categories for massive particles, i.e., (m, s_+) and (m, s_-) . To understand further why we have the two categories of the irreducible representations of the Poincaré group, let us study the relation between the two spin operators S_+^k and S_-^k , and their representations.

Let us first obtain the explicit representations of the two spin operators S_\pm^k in an arbitrary frame, which provides two inequivalent representations of the Poincaré group in a moving frame. The spin operators in a moving frame can be obtained from a Lorentz transformation from the particle rest frame in Eq. (19). Normally, the momentum of the particle changes for a Lorentz transformation. However, a rotation in the particle rest frame does not make the momentum of the particle changing. Thus, we can consider only a boost transformation from the particle rest frame. As is well-known, two successive non-collinear Lorentz boosts, equivalent to an effective rotation followed by an effective-single Lorentz boost, involve a nontrivial effect. However, such an effective rotation in the particle rest frame is not relevant to obtain the spin operators in a moving frame. Consequently, the explicit expressions of two spin operators for an arbitrary

reference frame are then determined by a single pure boost transformation (so-called standard Lorentz transformation) from the particle rest frame directly to the moving frame.

In the particle rest frame with zero momentum ($\mathbf{p} = 0$), for $s_+ = s_-$, the two spin operators in Eq. (19) are the same, i.e., $S_+^k(\mathbf{0}) = S_-^k(\mathbf{0})$. Then, we present the spin operators at the particle rest frame as the usual $\mathfrak{su}(2)$ operator, i.e., $S_\pm^k(\mathbf{0}) = \sigma^k/2$, where σ^k is the usual operator satisfying the $\mathfrak{su}(2)$ algebra. This representation is natural because the Pauli spin is the spin operator for a spin-1/2 massive particle in the nonrelativistic quantum mechanics. Thus, in order to obtain the explicit expressions of two spin operators in Eq. (19) in a moving frame, let us consider the standard Lorentz transformation $L(\mathbf{p})$, i.e., $L_0^0 = p^0/m$, $L_i^0 = p^i/m$, and $L_j^i = \delta_{ij} + p^i p^j / (m(p^0 + m))$ with the Kronecker-delta δ_{ij} . By using this standard Lorentz boost, the particle momentum $p^\mu = (p^0, \mathbf{p})$ is transformed as $p^\mu = L^\mu_\nu k^\nu$, where $k^\mu = (m, 0, 0, 0)$. In the particle rest frame, the PL vector becomes $W_{rest}^\mu = (0, m\boldsymbol{\sigma}/2)$, because $W_{rest}^0 = \frac{1}{2}\epsilon^{0ijk} J_{jk} P^i = 0$ and $S_\pm^k = W_{rest}^k/m = J^k/2$ is represented by $\sigma^k/2$. The PL vector in the moving frame, then transformed by the standard Lorentz transformation, is given as $W^0 = L^0_\mu W_{rest}^\mu = (\boldsymbol{\sigma} \cdot \mathbf{p})/2$ and $W^i = L^i_\mu W_{rest}^\mu = m\sigma^i/2 + p^i(\boldsymbol{\sigma} \cdot \mathbf{p})/(2(m + p^0))$. For an arbitrary reference frame with the momentum \mathbf{p} , the spin operators from Eq. (19) are obtained as

$$S_\pm^k(\mathbf{p}) = \frac{p^0}{2m}\sigma^k - \frac{p^k(\boldsymbol{\sigma} \cdot \mathbf{p})}{2m(p^0 + m)} \pm i\frac{1}{2m}(\boldsymbol{\sigma} \times \mathbf{p})^k. \quad (20)$$

As a result, we have obtained the explicit representations of the two spin operators $S_\pm^k(\mathbf{p})$ in an arbitrary reference frame, which provides the two inequivalent representations of the Poincaré group.

All representations of the Poincaré group are classified by the eigenvalues of two Casimir invariants, $P^\mu P_\mu = m^2$ and $S_\pm^k S_\pm^k = s_\pm(s_\pm + 1)$. The base spin states $\Psi_\pm(p, \lambda_\pm)$ of a representation space, on which the representation of the Poincaré group acts, are obtained by the following eigenvalue equations:

$$P^\mu \Psi_\pm(p, \lambda_\pm) = p^\mu \Psi_\pm(p, \lambda_\pm), \quad (21a)$$

$$S_\pm^k(\mathbf{p})\Psi_\pm(p, \lambda_\pm) = \lambda_\pm \Psi_\pm(p, \lambda_\pm), \quad (21b)$$

where p^μ is the specific momentum and $\lambda_\pm \in \{-s_\pm, -s_\pm + 1, \dots, s_\pm\}$ are the spin eigenvalues of the k -component of the spin operators, i.e., $S_\pm^k(\mathbf{p})$. Note that the eigenvalue equations in Eqs. (21a) and (21b) are valid for all reference frames. For $s_+ = s_- (\equiv s)$, i.e., $\lambda_+ = \lambda_- (\equiv \lambda)$, since $S_+^k(\mathbf{0}) = S_-^k(\mathbf{0})$ at the particle rest frame, the spin eigenstates are $\Psi_+(k, \lambda) = \Psi_-(k, \lambda) (\equiv \Psi(k, \lambda))$ and then Eq. (21b) becomes the spin eigenvalue equation: $S_\pm^k(\mathbf{0})\Psi(k, \lambda) = \lambda \Psi(k, \lambda)$. Then, $\Psi(k, \lambda)$ is a surely spin eigenstate because the particle has zero momentum ($\mathbf{p} = 0$) at the particle rest frame. Also,

since p^μ is the momentum of the particle in a specific frame moving with velocity $-\boldsymbol{\beta} = -\mathbf{p}\gamma^{-1}/m$ with respect to the particle rest frame, where the Lorentz factor is $\gamma = 1/\sqrt{1-\boldsymbol{\beta}^2}$ with the speed of light $c = 1$, $\Psi_\pm(p, \lambda)$ can be regarded as the spin states in the moving frame with the momentum \mathbf{p} . Furthermore, under parity (spatial inversion), since the momentum and the PL vectors transform as $(P^0, \mathbf{P}) \leftrightarrow (P^0, -\mathbf{P})$ and $(W^0, \mathbf{W}) \leftrightarrow (-W^0, \mathbf{W})$, respectively, the spin operator S_+^k transforms to the S_-^k and vice versa, i.e., $S_+^k \leftrightarrow S_-^k$ in Eqs. (19) and (20). Under parity, then, as the base states of the two inequivalent representations, the eigenstate $\Psi_+(p, \lambda)$ transforms to the $\Psi_-(p, \lambda)$ and vice versa, i.e., $\Psi_+(p, \lambda) \leftrightarrow \Psi_-(p, \lambda)$. This means that under parity, the two eigenstates $\Psi_+(p, \lambda)$ and $\Psi_-(p, \lambda)$, transformed from the eigenstate $\Psi(k, \lambda)$ at the particle rest frame, span the inequivalent representations reflecting the inequivalence of the two spin operators $S_+^k(\mathbf{p})$ and $S_-^k(\mathbf{p})$ in the moving frame.

In order to discuss clearer how the spin states $\Psi_+(p, \lambda)$ and $\Psi_-(p, \lambda)$ are related each other in the moving frame for $s_+ = s_-$, let us study transformation operators that can be used in obtaining $S_\pm^k(\mathbf{p})$ in Eq. (20) from $S_\pm^k(\mathbf{0})$. Actually, the explicit forms of the transformation operators $U_+[L(\mathbf{p})]$ and $U_-[L(\mathbf{p})]$ are respectively derived (see Appendix) as

$$U_+[L(\mathbf{p})] = \exp\left[+\frac{1}{2}\boldsymbol{\sigma} \cdot \boldsymbol{\xi}\right], \quad (22a)$$

$$U_-[L(\mathbf{p})] = \exp\left[-\frac{1}{2}\boldsymbol{\sigma} \cdot \boldsymbol{\xi}\right], \quad (22b)$$

where $\boldsymbol{\xi} = \hat{\mathbf{p}} \tanh^{-1}[|\mathbf{p}|/(p^0 + m)]$, and $U_\pm[L(\mathbf{p})]$ transform the spin operators $S_\pm^k(\mathbf{0})$ at the particle rest frame to the spin operators $S_\pm^k(\mathbf{p})$ in the moving frame:

$$S_+^k(\mathbf{p}) = U_+[L(\mathbf{p})]S_+^k(\mathbf{0})U_+^{-1}[L(\mathbf{p})], \quad (23a)$$

$$S_-^k(\mathbf{p}) = U_-[L(\mathbf{p})]S_-^k(\mathbf{0})U_-^{-1}[L(\mathbf{p})]. \quad (23b)$$

Then, the eigenstates $\Psi_+(p, \lambda)$ and $\Psi_-(p, \lambda)$ in the moving frame and $\Psi(k, \lambda)$ at the particle rest frame have the relations:

$$\Psi_+(p, \lambda) = U_+[L(\mathbf{p})]\Psi(k, \lambda), \quad (24a)$$

$$\Psi_-(p, \lambda) = U_-[L(\mathbf{p})]\Psi(k, \lambda). \quad (24b)$$

Hence, Eqs (23a) and (23b) ensure that the two eigenstates $\Psi_+(p, \lambda)$ and $\Psi_-(p, \lambda)$ are respectively transformed from the eigenstate $\Psi(k, \lambda)$ at the particle rest frame without changing the spin eigenvalue λ . It should be noted that the transformation operators $U_+[L(\mathbf{p})]$ and $U_-[L(\mathbf{p})]$ are the same as the left-handed and the right-handed representations of the boost transformation $L(\mathbf{p})$, respectively, in the (homogeneous) Lorentz group [25, 26]. Consequently, this fact implies that the representation spaces, whose base vectors are $\Psi_+(p, \lambda)$ and

$\Psi_-(p, \lambda)$, provide the left-handed and the right-handed representations in the Poincaré group. It should be also stressed that the spacetime symmetry naturally gives the two spin operators S_\pm^k , determining the handedness of the spin state, from the minimal conditions in (iii) and (iv) on the linear combination of PL vectors. The two spin operators are related by $S_+^k(-\mathbf{p}) = S_-^k(\mathbf{p})$ in Eq. (20) because $U_+[L(-\mathbf{p})] = U_-[L(\mathbf{p})]$. Eqs (23a) and (23b) imply that the two spin operators are Lorentz-covariant.

In addition, let us consider the representation space for the Poincaré group extended by parity. As we discussed, under parity, the base states $\Psi_+(p, \lambda)$ and $\Psi_-(p, \lambda)$ are interchanged each other. This requires that the representation space for the Poincaré group extended by parity should contain all states of the two types of $\Psi_+(p, \lambda)$ and $\Psi_-(p, \lambda)$. Then, the naturally generalized representations, including both kinds of states $\Psi_+(p, \lambda)$ and $\Psi_-(p, \lambda)$, are the tensor product of the left-handed and the right-handed representations, i.e., (s_+, s_-) representations. The parity operation, however, does not allow all the tensor product representations. As a consequence, in the Poincaré group extended by parity, possible representations are nonchiral (s, s) representations and direct-sum $(s_+, s_-) \oplus (s_-, s_+)$ representations.

(viii) *Each of the two spin three-vector operators generates a little group of the Poincaré group.* Satisfying the $\mathfrak{su}(2)$ algebra, each of S_\pm^k generates a SU(2) group. The elements of the SU(2) group can be denoted by $\mathcal{D}_\pm(\theta_\pm^k) = \exp\left[\frac{i}{2}\theta_\pm^k S_\pm^k\right]$ with a finite parameter θ_\pm^k of the rotation group. The group elements $\mathcal{D}_\pm(\theta_\pm^k)$ do not change the momentum of a particle because

$$\mathcal{D}_\pm(\theta_\pm^k)P^\mu\mathcal{D}_\pm^{-1}(\theta_\pm^k) = P^\mu, \quad (25)$$

which is guaranteed by $[S_\pm^k, P^\mu] = 0$. As is known, the subgroup of the Lorentz group that does not change the momentum of a particle is called the Wigner little group. To complete our argument that the group composed of every element $\mathcal{D}_\pm(\theta_\pm^k)$ is a Wigner little group, we will then show that the action of $\mathcal{D}_\pm(\theta_\pm^k)$ on the spin states is represented by Lorentz transformations.

From now on, when every argument for the left-handed representation denoted by the subscript $+$ can be equally applied to the right-handed representation $-$, we will freely use only the subscript $+$ instead of \pm for simplicity. Since the group element $\mathcal{D}_+(\theta_+^k)$ is generated by the spin operators S_+^k and S_-^k , the representation space is composed of the eigenstates $\Psi_+(p, \lambda)$ and $\Psi_-(p, \lambda)$, respectively. To study the case that gives the Wigner little group in these representation spaces, let us first consider the successive non-collinear Lorentz transformations, transforming the particle back to its rest frame, from the particle rest frame. All such Lorentz transformations are reduced to the following transformations:

$$R(\Lambda, \mathbf{p}) = L^{-1}(\Lambda\mathbf{p})\Lambda L(\mathbf{p}), \quad (26)$$

where $L^{-1}(\Lambda \mathbf{p})$ is the inverse of $L(\Lambda \mathbf{p})$. The Lorentz transformations Λ and $L(\mathbf{p})$ are non-collinear, which transforms the rest momentum k^μ to $q^\mu = \Lambda^\mu_\nu L(\mathbf{p})^\nu_\rho k^\rho$. $L(\Lambda \mathbf{p})$ is the standard transformation giving $q^\mu = L(\Lambda \mathbf{p})^\mu_\rho k^\rho$ with $L(\Lambda \mathbf{p}) \neq \Lambda L(\mathbf{p})$. The two successive non-collinear Lorentz transformations $\Lambda L(\mathbf{p})$ are equivalent to the rotation followed by the standard Lorentz transformation $L(\Lambda \mathbf{p})$, i.e., $\Lambda L(\mathbf{p}) = L(\Lambda \mathbf{p})R(\Lambda, \mathbf{p})$. Hence, the operation of $L^{-1}(\Lambda \mathbf{p})\Lambda L(\mathbf{p})$ becomes the pure rotation $R(\Lambda, \mathbf{p})$ in Eq. (26), which does not change the rest momentum k^μ of the particle. In the left-handed and the right-handed representations, the transformation acting on the spin state space of the particle rest frame for such a rotational transformation $R(\Lambda, \mathbf{p})$ in Eq. (26) becomes

$$\tilde{\mathcal{D}}_\pm(R(\Lambda, \mathbf{p})) = U_\pm^{-1}[L(\Lambda \mathbf{p})]U_\pm[\Lambda]U_\pm[L(\mathbf{p})] \quad (27a)$$

$$= U_\pm[L^{-1}(\Lambda \mathbf{p})\Lambda L(\mathbf{p})], \quad (27b)$$

with using the group laws $U[A]U[B]U[C] = U[ABC]$ and $U^{-1}[A] = U[A^{-1}]$. Thus, the $\tilde{\mathcal{D}}_\pm(R(\Lambda, \mathbf{p}))$ represents a rotation in the spin state space of the particle rest frame without changing the eigenvalue of momentum. It is well-known that for a pure rotation, the left-handed and the right-handed representations $\tilde{\mathcal{D}}_+(\Lambda, \mathbf{p})$ and $\tilde{\mathcal{D}}_-(\Lambda, \mathbf{p})$ are the same each other and reduce to the usual rotation matrix [26].

As is mentioned in (vii), for $s_+ = s_-$, the two spin operators S_\pm^k are represented by the same operator σ^k at the particle rest frame. Their eigenstates are then the same as $\Psi(k, \lambda)$. Therefore, in the representation space composed of $\Psi(k, \lambda)$, the representation of $\tilde{\mathcal{D}}_\pm(R(\Lambda, \mathbf{p}))$ becomes $\exp[\frac{i}{2}\phi_\pm^k(\Lambda, \mathbf{p})\sigma^k]$, where the rotation angle parameters $\phi_\pm^k(\Lambda, \mathbf{p})$ are determined by the Lorentz transformations Λ and $L(\mathbf{p})$ through the rotation $R(\Lambda, \mathbf{p})$. Such a rotation has nothing to do with the handedness. The rotation angles $\phi_\pm^k(\Lambda, \mathbf{p})$ do not depend on the handedness. This fact allows us denoting $\phi_\pm^k(\Lambda, \mathbf{p})$ as just $\phi^k(\Lambda, \mathbf{p})$. It is shown that the representation of $\tilde{\mathcal{D}}_\pm(R(\Lambda, \mathbf{p}))$ is nothing but the element $\mathcal{D}_\pm(\theta^k)$ generated by the rest spin operator σ^k with the rotation angle $\phi^k(\Lambda, \mathbf{p}) = \theta_+^k = \theta_-^k (\equiv \theta^k)$.

In an arbitrary reference frame, where the particle has a momentum $\tilde{\mathbf{p}}$, the spin states are represented by $\Psi_\pm(\tilde{\mathbf{p}}, \lambda) = U_\pm[L(\tilde{\mathbf{p}})]\Psi(k, \lambda)$ with a standard Lorentz transformation $U_\pm[L(\tilde{\mathbf{p}})]$. For the arbitrary reference frame, the element $\mathcal{D}_\pm(\theta^k) = \exp[\frac{i}{2}\theta^k\sigma^k]$ are transformed with using Eqs. (23a) and (23b):

$$U_\pm[L(\tilde{\mathbf{p}})]\exp\left[\frac{i}{2}\theta^k\sigma^k\right]U_\pm^{-1}[L(\tilde{\mathbf{p}})] = \exp\left[\frac{i}{2}\theta^k S_\pm^k(\tilde{\mathbf{p}})\right]. \quad (28)$$

The $\exp[\frac{i}{2}\theta^k S_\pm^k(\tilde{\mathbf{p}})]$ is the element $\mathcal{D}_\pm(\theta^k)$ of SU(2) group generated by $S_\pm^k(\tilde{\mathbf{p}})$ in the arbitrary reference frame. The group element $\mathcal{D}_\pm(\theta^k)$ is determined solely by the representations of the Lorentz transformations, $U_+[\Lambda]$, $U_+[L(\mathbf{p})]$, $U_+[L(\Lambda \mathbf{p})]$, and $U_+[L(\tilde{\mathbf{p}})]$ without changing

the momentum. As a result, the two SU(2) groups with the elements $\mathcal{D}_\pm(\theta^k)$ generated by the two spin operators $S_\pm^k(\tilde{\mathbf{p}})$ are Wigner little groups. Their elements rotate the spin states without changing the momentum eigenvalue of the spin state. The rotation angle parameters θ_\pm^k are determined from the detailed informations of Lorentz transformations which give the particle momentum p^μ . One of the two Wigner little groups is chosen to represent the Poincaré group.

(ix) *An arbitrary Lorentz transformation on the left-handed and the right-handed spin states gives the same Wigner rotation.* Let us discuss the effects of the little group elements $\mathcal{D}_\pm(\theta^k) = \exp[\frac{i}{2}\theta^k S_\pm^k(\mathbf{p})]$ on the spin states $\Psi_\pm(p, \lambda)$. By using Eq. (28) and $\Psi_\pm(p, \lambda) = U_\pm[L(\mathbf{p})]\Psi(k, \lambda)$, one can find the equality:

$$\exp\left[\frac{i}{2}\theta^k S_\pm^k\right]\Psi_\pm(p, \lambda) = U_\pm[L(\mathbf{p})]\exp\left[\frac{i}{2}\theta^k \sigma^k\right]\Psi(k, \lambda), \quad (29)$$

Note that for both rotations on the spin states in the particle rest frame and in the moving frame, the angle parameter θ^k does not change. As is shown in (vii), the spin eigenvalues of S_\pm^k and σ^k are the same, and S_\pm^k and σ^k are the generators for rotations about k -th space coordinate axis. This implies that all three rotations, respectively generated by the spin operators S_+^k , S_-^k , and σ^k , are equivalent. Consequently, the two spin operators S_\pm^k in an arbitrary frame provide the same Wigner rotation.

Actually, a general Lorentz transformation on the left-handed and the right-handed representations can be represented with the action of our little groups. Let us consider a Lorentz transformation $U_\pm[\Lambda]$ on the spin states $\Psi_\pm(p, \lambda)$:

$$U_\pm[\Lambda]\Psi_\pm(p, \lambda) = U_\pm[\Lambda]U_\pm[L(\mathbf{p})]\Psi(k, \lambda) \quad (30a)$$

$$= U_\pm[L(\Lambda \mathbf{p})]\tilde{\mathcal{D}}_\pm(\Lambda, \mathbf{p})\Psi(k, \lambda). \quad (30b)$$

With the usual Wigner rotation in the $(2s + 1)$ -dimensional representation, Eq. (30b) can be given as

$$U_\pm[\Lambda]\Psi_\pm(p, \lambda) = \sum_{\lambda'} D_{\lambda'\lambda}(\Lambda, \mathbf{p})\Psi_\pm(\Lambda p, \lambda'), \quad (31)$$

where $D_{\lambda'\lambda}(\Lambda, \mathbf{p})$ is the usual $(2s + 1)$ -dimensional rotation matrix. It is shown that the spin states in both the left-handed and the right-handed representations undergo the same Wigner rotation for general Lorentz transformations. Therefore, the handedness is determined by only a standard Lorentz transformation on the spin states. Because of the Wigner rotation of the spin states, the consideration of a fixed spin is not physically meaningful in relativistic situation [27].

(x) *Derivation of the covariant Dirac equation by using the transformation property of a covariant parity operator on the direct sum $(1/2, 0) \oplus (0, 1/2)$ representation.* As the most successful equation in describing relativistic massive particles, the Dirac equation is given in the

direct sum $(1/2, 0) \oplus (0, 1/2)$ representation for the spin-1/2 massive particles. In the Dirac Hamiltonian and the covariant Dirac equation, the Dirac matrices consist of only the Pauli spin matrices as the submatrices. Also, as is shown in (vii), the Pauli spin matrices are responsible merely for the representation of the Dirac particle at the particle rest frame. These facts make it unclear how a relativistic spin rather than the Pauli spin affects the representation of the Dirac particles through the Dirac equation.

Let us then discuss how the property of the $(1/2, 0) \oplus (0, 1/2)$ representation of the spin operators S_{\pm}^k connects to the covariant Dirac equation. As an irreducible single-spin representation for $s = 1/2$, the direct sum $(1/2, 0) \oplus (0, 1/2)$ representation space is composed of a linear combination of the states

$$\psi(p, \lambda) = \begin{pmatrix} \Psi_+(p, \lambda) \\ \Psi_-(p, \lambda) \end{pmatrix}, \quad (32)$$

where $\Psi_+(p, \lambda)$ and $\Psi_-(p, \lambda)$ are the two-dimensional spin eigenstates of S_+^k and S_-^k with the eigenvalues $\lambda \in \{-1/2, 1/2\}$, respectively, as shown in (vii). Since $\Psi_+(-\mathbf{p}, \lambda) = \Psi_- (\mathbf{p}, \lambda)$, the parity operation \mathcal{P} on the state $\Psi(p, \lambda)$ in the $(1/2, 0) \oplus (0, 1/2)$ representation space can be represented by

$$\mathcal{P} \psi(p, \lambda) = \begin{pmatrix} \Psi_-(p, \lambda) \\ \Psi_+(p, \lambda) \end{pmatrix} = \gamma^0 \psi(p, \lambda), \quad (33)$$

where $\gamma^0 = \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix}$ with the two-dimensional identity matrix \mathbb{I}_2 . The state $\psi(p, \lambda)$ is then an irreducible representation of the parity-extended Poincaré group for spin-1/2 massive particles. However, the γ^0 is not a covariant representation of the parity operation \mathcal{P} . Using the transformation properties of the spin states and the relativistic spin operators S_{\pm}^k under parity allows us to construct the covariant parity operator. The covariant form of the parity operator is represented as

$$\mathcal{P} \psi(p, \lambda) = \frac{1}{m} \left[p^0 + 2 \begin{pmatrix} {}^*S_{0\mu;+} p^\mu & 0 \\ 0 & -{}^*S_{0\mu;-} p^\mu \end{pmatrix} \right] \psi(p, \lambda), \quad (34)$$

where the spin dual tensors ${}^*S_{\nu\mu;\pm}$ are antisymmetric and their components are ${}^*S_{00;\pm} = 0$ and ${}^*S_{0k;\pm} = S_{\pm}^k$ in Eq. (13). This covariant representation transforms as 0-th component of a four-vector under a Lorentz transformation. Note that the operators ${}^*S_{0\mu;\pm} p^\mu = S_{\pm}^k p^k$ become W^0 from Eq. (19) and then in the moving frame, $W^0 = \boldsymbol{\sigma}/2 \cdot \mathbf{p}$, i.e., $S_{\pm}^k p^k = \sigma^k p^k/2$. Thus, the covariant parity operator can be rewritten as $(p^0 + \sigma^k p^k \gamma^5)/m$, where $\gamma^5 = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}$. By using $\Psi_{\pm}(p, \lambda) = U_{\pm}[L(p)]\Psi(k, \lambda)$ in Eq. (24a) and the explicit forms of $U_{\pm}[L(p)]$ in Eq. (22b), one can easily confirm that the operation of

$(p^0 + \sigma^k p^k \gamma^5)/m$ is equivalent to the operation of γ^0 :

$$\frac{1}{m} [p^0 + \sigma^k p^k \gamma^5] \begin{pmatrix} \Psi_+(p, \lambda) \\ \Psi_-(p, \lambda) \end{pmatrix} = \begin{pmatrix} \Psi_-(p, \lambda) \\ \Psi_+(p, \lambda) \end{pmatrix}. \quad (35)$$

Multiplying both sides of Eq. (35) by γ^0 and rearranging the equation reproduce the covariant Dirac equation as the usual form $(\gamma^\mu p_\mu - m)\psi(p, \lambda) = 0$. Consequently, this rederivation of the covariant Dirac equation shows a manifestation of the relativistic spins.

In deriving the covariant Dirac equation, additionally, the noticeable crucial property of the spins is their projections onto the particle momentum, i.e., the projection of the relativistic spins is equivalent to the one of the rest spin, i.e., $\mathbf{S}_{\pm} \cdot \mathbf{p} = \boldsymbol{\sigma}/2 \cdot \mathbf{p}$ and $(\mathbf{S}_{\pm} \cdot \mathbf{p})^2 = \mathbf{p}^2/4$. On the one hand, this shows that the appearance of the Pauli matrices in the Dirac equation is a natural consequence originated from the relativistic spin operators S_{\pm}^k . On the other hand, the equivalence makes the use of the helicity operator $\boldsymbol{\sigma}/2 \cdot \hat{\mathbf{p}}$ [25] justified for relativistic spin-1/2 particles, where $\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$. Some general remarks on the helicity will be given for relativistic Dirac-like particle with spin s in the following (xiii).

(xi) *The direct sum $(s, 0) \oplus (0, s)$ representation provides a parity-conserving theory like the Dirac theory and gives a unique axial spin-three vector operator.* As one of the parity-conserving representations in the Poincaré group extended by parity, let us consider the direct-sum $(s, 0) \oplus (0, s)$ representation of a single spin- s massive particle. Actually, one can construct directly a spin operator by using the left-handed and the right-handed spin operators in Eq. (19). In the $2(2s+1)$ -dimensional representation, the unique spin operator for the spin- s massive particle is given as

$$\mathcal{S}^k = S_+^k \oplus S_-^k = \begin{pmatrix} S_+^k & 0 \\ 0 & S_-^k \end{pmatrix} \quad (36a)$$

$$= \frac{\mathcal{I}}{m^2} (P^0 W^k - P^k W^0) + i \frac{\Upsilon^5}{m^2} \epsilon_{kml} P^l W^m, \quad (36b)$$

with the $2(2s+1)$ -dimensional identity matrix $\mathcal{I} = \mathbb{I}_{2(2s+1)}$ and $\Upsilon^5 = \begin{pmatrix} \mathbb{I}_{2s+1} & 0 \\ 0 & -\mathbb{I}_{2s+1} \end{pmatrix}$. One can show clearly that the \mathcal{S} in Eq. (36b) satisfies the $\mathfrak{su}(2)$ algebra and the Lorentz-transformation properties as the tensorial requirement. Obviously, the \mathcal{S}^k provides the same Casimir operator $\mathcal{S}^k \mathcal{S}^k = -W^\mu W_\mu/m^2$ in the Poincaré group. The Casimir operator $\mathcal{S}^k \mathcal{S}^k$ has the eigenvalues $s(s+1)$. As is expected, the direct-sum $(s, 0) \oplus (0, s)$ representation in the Poincaré group extended by parity is shown to be the irreducible representation for a massive particle with the mass m and the spin s , which is labelled by the eigenvalues of $\{P^\mu, \mathcal{S}^k\}$.

Based on the discussions in (vii), the eigenvalue equations for the $(s, 0) \oplus (0, s)$ representation can be written

as

$$P^\mu \psi(p, \lambda) = p^\mu \psi(p, \lambda), \quad (37a)$$

$$\mathcal{S}^k(\mathbf{p})\psi(p, \lambda) = \lambda \psi(p, \lambda), \quad (37b)$$

where with $\lambda \in \{-s, -s+1, \dots, s\}$, the eigenstates are given by

$$\psi(p, \lambda) = \begin{pmatrix} \Psi_+(p, \lambda) \\ \Psi_-(p, \lambda) \end{pmatrix} = U[L(\mathbf{p})]\psi(k, \lambda). \quad (37c)$$

Here, the $2(2s+1)$ -dimensional standard Lorentz transformation is given in the direct-sum representation as $U[L(\mathbf{p})] = \begin{pmatrix} U_+[L(\mathbf{p})] & 0 \\ 0 & U_-[L(\mathbf{p})] \end{pmatrix}$. Under parity, the left (right)-handed states transform into the right (left)-handed states, i.e., $\Psi_+(p, \lambda) \leftrightarrow \Psi_-(p, \lambda)$ and thus the eigenstate $\psi(p, \lambda)$ transforms to $\psi(p, \lambda) \rightarrow \Gamma^0 \psi(p, \lambda)$ with $\Gamma^0 = \begin{pmatrix} 0 & \mathbb{I}_{2s+1} \\ \mathbb{I}_{2s+1} & 0 \end{pmatrix}$. The Lorentz-invariant scalar product can be defined as $\bar{\psi}\psi$, where $\bar{\psi} = \psi^\dagger \Gamma^0$. The spin three-vector \mathcal{S}^k in Eq. (36b) is axial for massive particles in the sense that under parity, the expectation value of $\bar{\psi}\mathcal{S}^k\psi$ is invariant, i.e., an axial three-vector because the eigenstate ψ changes as $\psi \rightarrow \Gamma^0\psi$ and $\mathcal{S}^k \rightarrow \Gamma^0\mathcal{S}^k\Gamma^0$ due to $\mathbf{S}_+ \leftrightarrow \mathbf{S}_-$.

Such a direct-sum $(s, 0) \oplus (0, s)$ representation has been used to describe a spin- s massive particle in extending the spin-1/2 Dirac equation to the spin- s Dirac-like equation by Weinberg [26]. In fact, our $2(2s+1)$ -dimensional state $\psi(p, \lambda)$ satisfies the Dirac-like equation describing a spin- s massive Dirac-like particle:

$$[\gamma^{\mu_1\mu_2\cdots\mu_{2s}} p_{\mu_1} p_{\mu_2} \cdots p_{\mu_{2s}} + (-m)^{2s}] \psi(p, \lambda) = 0, \quad (38a)$$

where the generalized γ matrices are defined as

$$\gamma^{\mu_1\mu_2\cdots\mu_{2s}} = \begin{pmatrix} 0 & \sigma^{\mu_1\mu_2\cdots\mu_{2s}} \\ \bar{\sigma}^{\mu_1\mu_2\cdots\mu_{2s}} & 0 \end{pmatrix} \quad (38b)$$

with $\bar{\sigma}^{\mu_1\mu_2\cdots\mu_{2s}} = (-1)^n \sigma^{\mu_1\mu_2\cdots\mu_{2s}}$ and the number n of spacelike indexes in μ 's [26]. The $2s$ -rank tensor $\sigma^{\mu_1\mu_2\cdots\mu_{2s}}$ is symmetric and traceless in all μ 's. The Dirac-like equation in Eq. (38a) has a chiral symmetry. For spin-1/2, as is known, Eq. (38a) reduces to the Dirac equation.

(xii) *The spin current is conserved by itself for a massive Dirac-like particle: The axial spin three-vector in an arbitrary frame is a good observable.* For the massive Dirac-like particle in Eq. (38a), a conserved spin current can be defined as

$$(J_S^\mu)^k = i^{2s-1} [\bar{\Phi}(x) \gamma^{\mu\mu_1\cdots\mu_{2s-1}} \partial_{\mu_1} \cdots \partial_{\mu_{2s-1}} \mathcal{S}^k \Phi(x) - \partial_{\mu_1} \bar{\Phi}(x) \gamma^{\mu\mu_1\cdots\mu_{2s-1}} \partial_{\mu_2} \cdots \partial_{\mu_{2s-1}} \mathcal{S}^k \Phi(x) + \cdots + (-)^{2s+1} \partial_{\mu_1} \cdots \partial_{\mu_{2s-1}} \bar{\Phi}(x) \gamma^{\mu\mu_1\cdots\mu_{2s-1}} \mathcal{S}^k \Phi(x)], \quad (39)$$

where $\Phi(x)$ is the solution of the Dirac-like equation in x -representation, i.e., $\Phi(x) =$

$\sum_\lambda \int d^4p [a(p, \lambda) e^{-ip^\mu x_\mu} \Phi_+(p, \lambda) + b(p, \lambda) e^{ip^\mu x_\mu} \Phi_-(p, \lambda)]$. Here, d^4p is the Lorentz invariant integration measure, and the coefficients $a(p, \lambda)$ and $b(p, \lambda)$ are a function of p and λ . $\Phi_\pm(p, \lambda) = \Psi_+(p, \lambda) \pm \Psi_-(p, \lambda)$ are the positive and the negative energy eigenstates, respectively. For a conserved spin current, then, $\partial_\mu (J_S^\mu)^k = 0$ requires

$$[\mathcal{S}^k, \gamma^{\mu_1\cdots\mu_{2s}} p_{\mu_1} \cdots p_{\mu_{2s}}] = 0. \quad (40)$$

For the massive Dirac-like particles, the fact that $[\mathcal{S}_{rest}^k, m\Gamma^0] = 0$ at the particle rest frame allows us to prove the requirement in Eq. (40) by considering a Lorentz transformation $U[\Lambda] = U_+[\Lambda] \oplus U_-[\Lambda]$ transforming from the particle rest frame to an arbitrary reference frame, where \mathcal{S}_{rest}^k is the spin operator at the particle rest frame in Eq. (36b). Here we use $\Gamma^0 = -i^{-2s} \gamma^{00\cdots 0}$ [26]. By using $\mathcal{S}^k = U[\Lambda] \mathcal{S}_{rest}^k U^{-1}[\Lambda]$, $p_\mu = \Lambda_\mu^{\nu} k_\nu$, and $U[\Lambda] \gamma^{\mu_1\cdots\mu_{2s}} U^{-1}[\Lambda] = \gamma^{\nu_1\cdots\nu_{2s}} \Lambda_{\nu_1}^{\mu_1} \cdots \Lambda_{\nu_{2s}}^{\mu_{2s}}$, we see $[\mathcal{S}^k, \gamma^{\mu_1\cdots\mu_{2s}} p_{\mu_1} \cdots p_{\mu_{2s}}] = U[\Lambda] [\mathcal{S}_{rest}^k, \gamma^{\nu_1\cdots\nu_{2s}} k_{\nu_1} \cdots k_{\nu_{2s}}] U^{-1}[\Lambda] = 0$. Then, the corresponding conserved charges are the expectation values of the spin. For spin 1/2, Eq. (39) reduces to the spin current given by the Poincare symmetry according to the Noether theorem.

Consequently, our spin operator in Eq. (36b) satisfies the spin current conservation for a massive Dirac-like particle, which shows that the \mathcal{S}^k in Eq. (36b) is a good observable, in the contrast to the non-conserving spin current in spintronics [5]. It should be also noted that like the case of non-relativistic systems where one can specify a given energy state by the projection of spin along the z -axis (namely, by the eigenvalue of S_z), in the relativistic case such a specification is useful since spin is a constant of motion.

Remarks.— (xiii) For the single spin- s massive Dirac-like particles, the helicity defined as the projection of the particle spin at the direction of motion, i.e., $\mathfrak{H} = \mathcal{S} \cdot \hat{\mathbf{p}}$ satisfies

$$[\mathfrak{H}, \gamma^{\mu_1\cdots\mu_{2s}} p_{\mu_1} \cdots p_{\mu_{2s}}] = 0. \quad (41)$$

Equations (41) implies that the spin states satisfying the Dirac-like equation in Eq. (38a) can also be the eigenstate of the helicity operator. However, this does not mean that the helicity \mathfrak{H} is a good quantum number like the spin, as is shown in (xii). The helicity operator does not have a well-defined Lorentz transformation property so that the expectation value of the helicity operator changes not covariantly among observers in different reference frames, in the contrast to the spin.

(xiv) Our spin operator \mathcal{S}^k in Eq. (36b) in the direct sum $(s, 0) \oplus (0, s)$ representation is valid for any massive particle with an integer or half-integer spin. For a spin-1/2 massive particle satisfying the Dirac equation, our spin operator becomes the Chakrabarti spin operator [14]. For a positive-energy particle of the spin-1/2 massive Dirac particle, also, our spin operator is equivalent to

the Pryce spin operator [10] and the Foldy-Wouthuysen spin operator [12], as shown in Ref. [20].

(xv) The spin information of Dirac-like massive particles described by our spin operator S^k in Eq. (36b) can be properly delivered by the spin reduced density matrix obtained from the partial tracing over momentum degrees of freedom in the Fold-Wouthuysen representation because the separation of momentum degrees of freedom and spin degrees of freedom has been manifested in the Fold-Wouthuysen representation [12] for the spin-1/2 massive Dirac particle (e.g., electron). To deal with changes of the spin information under Lorentz transformations, then, the Foldy-Wouthuysen representation can provide a more convenient way [28].

APPENDIX

The spin operator $S_{\pm}^k(\mathbf{p})$ in Eq. (20) have been obtained from the S_{\pm}^k in Eq. (19) by using the standard boost Lorentz transform $L(\mathbf{p})$ from the rest frame to the moving frame with the momentum \mathbf{p} . This implies that the spin operator $S_{\pm}^k(\mathbf{p})$ in Eq. (20) can be reexpressed as a usual transformation form, i.e., $U_{\pm} S_{\pm}^k(\mathbf{0}) U_{\pm}^{-1}$ in terms of the rest spin operator $S_{\pm}^k(\mathbf{0})$ with a transformation operator U . Prior to manipulate the right-handed side of Eq. (20), let us define $\cosh \frac{\xi}{2} = \sqrt{\frac{p^0+m}{2m}}$ and $\sinh \frac{\xi}{2} = \sqrt{\frac{p^0-m}{2m}}$ with $(p^0)^2 = |\mathbf{p}|^2 + m^2$. One can then manipulate the right-handed side of $S_{+}^k(\mathbf{p})$ in Eq. (20) such as

$$S_{+}^k(\mathbf{p}) = \frac{\sigma^k}{2} + \sinh \xi A^k + (\cosh \xi - 1) B^k \quad (42a)$$

$$= \frac{\sigma^k}{2} + \sum_{n=1} \left[\frac{\xi^{2n-1}}{(2n-1)!} A^k + \frac{\xi^{2n}}{2n!} B^k \right], \quad (42b)$$

where $A^k = (\boldsymbol{\sigma} \times \hat{\mathbf{p}})^k / 2$ and $B^k = \sigma^k / 2 - \hat{p}^k (\boldsymbol{\sigma} \cdot \hat{\mathbf{p}}) / 2$. One can notice that Eq. (42b) can be expressed as a form of $e^X Y e^{-X} = Y + \sum_{n=1} X_n / n!$ with $X_{n+1} = \frac{1}{n+1} [X, X_n]$ and $X_1 = [X, Y]$ in the Baker-Hausdorff formula because the first term can be $Y = \sigma^k / 2$. Then, let us work out an explicit form of the operator X by assuming the transformation operator as $U_{+} = \exp[X]$, where $X = f(\boldsymbol{\sigma})$ is a function of the rest spin operator $\boldsymbol{\sigma}$ with the given momentum p^{μ} in a moving frame. In terms of the function $f(\boldsymbol{\sigma})$, the recursive relation is given as $X_{n+1} = \frac{1}{n+1} [f(\boldsymbol{\sigma}), X_n]$ with $X_1 = [f(\boldsymbol{\sigma}), \sigma^k / 2]$. Comparing with Eq. (42b), we have the two relations $X_{2n-1} = \frac{\xi^{2n-1}}{(2n-1)!} (\boldsymbol{\sigma} / 2 \times \hat{\mathbf{p}})^k$ and $X_{2n} = \frac{\xi^{2n}}{2n!} (\sigma^k / 2 - \hat{p}^k (\boldsymbol{\sigma} / 2 \cdot \hat{\mathbf{p}}))$. In determining the function $f(\boldsymbol{\sigma})$, thus, we have the two conditions $X_1 = [f(\boldsymbol{\sigma}), \sigma^k / 2] = \xi (\boldsymbol{\sigma} / 2 \times \hat{\mathbf{p}})^k$ and $\frac{1}{2n+1} [f(\boldsymbol{\sigma}), (\boldsymbol{\sigma} / 2 \times \hat{\mathbf{p}})^k] = \frac{\xi}{2n} (\sigma^k / 2 - \hat{p}^k (\boldsymbol{\sigma} / 2 \cdot \hat{\mathbf{p}}))$. By using the $\mathfrak{su}(2)$ algebra $[\sigma^i, \sigma^j] = 2i\epsilon_{ijk} \sigma^k$, we see $(\boldsymbol{\sigma} \times \hat{\mathbf{p}})^k = [\sigma^j, \sigma^k] \hat{p}^j / 2$ and then find $f(\boldsymbol{\sigma}) = \xi \sigma^j \hat{p}^j / 2$.

By putting the function $f(\boldsymbol{\sigma}) = \xi \sigma^j \hat{p}^j / 2$ into the second condition, one can find that the equality of the second condition holds. The $S_{+}^k(\mathbf{p})$ in Eq. (20) is reexpressed as

$$S_{+}(\mathbf{p}) = \exp \left[\frac{1}{2} \boldsymbol{\sigma} \cdot \boldsymbol{\xi} \right] \left(\frac{\sigma^k}{2} \right) \exp \left[-\frac{1}{2} \boldsymbol{\sigma} \cdot \boldsymbol{\xi} \right].$$

Consequently, the spin operator $S_{+}^k(\mathbf{p})$ is the standard boost Lorentz transformation of the rest spin operator $S_{+}^k(\mathbf{0})$ and the Lorentz transformation operator for the spin operators is defined as

$$U_{+} = \exp \left[\frac{\boldsymbol{\sigma}}{2} \cdot \boldsymbol{\xi} \right].$$

Similarly, we also obtain $U_{-} = \exp[-\boldsymbol{\sigma} / 2 \cdot \boldsymbol{\xi}]$ from the $S_{-}^k(\mathbf{p})$ in Eq. (20).

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* Electronic address: tschoi@swu.ac.kr

† Electronic address: sycho@cqu.edu.cn

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